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MINIMAL MANEUVERS OF HIGH-PERFORMANCE
AIRCRAFT IN A VERTICAL PLANE

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SUMMARY

A general theory is presented for analyzing minimal maneuvers of high-performance aircraft in a vertical plane where the earth is assumed flat and the gravitational field uniform.

The indirect methods of the calculus of variations are used and the variational problem formulated as a problem of Bolza type. The latter consists of extremizing the sum of a line integral and of a functional expression which depends upon the end values of the generalized coordinates of the aircraft. Problems of either the Lagrange type or the Mayer type are studied as particular cases.

For the general problem of simultaneously optimizing the angle-of-attack program and the thrust program, solutions in a closed form are not possible. Thus, the integration of the set of Euler equations and constraining equations is to be performed by approximate methods. In view of the fact that the variational problems of engineering interest are of the boundary-value type (i.e., problems with conditions prescribed in part at the initial point and in part at the final point), the use of trial and error procedures is in order.

Under particular assumptions, however, expressions in a closed form can be derived for the optimizing condition. As an example, the brachistocronic climbing technique of a rocket-powered aircraft is readily computed if one neglects the induced drag with respect to the zero-lift drag. As another example, the climbing technique of minimum time or of minimum fuel consumption for a turbojet aircraft is also readily computed by neglecting centripetal accelerations. For these problems the relationship between altitude and Mach number is explicitly calculated, thus bypassing the tedious energy-height method.

To complete the paper, and to establish a link between calculus of variations and ordinary theory of maxima and minima, some quasi-steady problems (i.e., problems in which the acceleration terms are neglected) of Mayer type are considered. In particular, the maximum range and the maximum endurance of a turbojet aircraft are investigated for constant

flight altitude or for constant engine control parameter. Simple equations are derived for the ratio of induced drag to zero-lift drag and for the optimizing condition. These relations constitute an important extension of the well-known results of the low-speed flight theory to the case where an arbitrary interdependence between zero-lift drag coefficient, induced drag factor, thrust, specific fuel consumption, and Mach number is considered.

INTRODUCTION

Several new problems of applied mathematics have arisen in the analysis of trajectories of high-speed aircraft which cannot be handled by using conventional methods of performance analysis. Typical among these problems is the determination of the optimum climbing technique between one altitude and another.

In the years preceding the second World War, it was customary to investigate the flight performances of conventional aircraft by assuming that the motion of the center of gravity is locally straight and uniform. With the above hypothesis substantial simplifications were made possible, leading to simple analytical relationships of great usefulness for design purposes.

For the case of a jet-propelled aircraft[1] it becomes important to account for the inertia terms because of the rapid variation of the vector velocity with the time. Thus, the analysis of the optimum flight performances shifts from the domain of the ordinary theory of maxima and minima into the realm of the calculus of variations.

Simplified Climbing Problem for Turbojet Aircraft

The problem of the optimum climbing program for turbojet aircraft attracted considerable interest in the years immediately following the second World War. The initial investigations were based on the assumption

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[1]Some of the essential differences between conventional aircraft and jet-propelled aircraft can be illustrated by means of the following comparison which refers to climbing flight: for a piston-engine aircraft which must be transferred from sea level and take-off speed to $M = 0.5$ and $h = 20,000$ feet the variation in kinetic energy per unit mass is about one-fifth the variation in potential energy; on the other hand, for a jet-propelled vehicle which must be transferred from sea level and take-off speed to $M = 2.5$ and $h = 50,000$ feet the variation in kinetic energy is about twice the variation in potential energy.

that the mass of the aircraft is ideally a constant and the centripetal component of the acceleration is disregarded, with only the tangential component being accounted for.

Thus, approximate solutions were detected (refs. 1 and 2) by Miele, whose analytical method (based on Green's theorem) avoided the use of indirect variational procedures. The calculus of variations was also avoided in a paper by Lush (ref. 3), with an intuitive graphical method based on the concept of energy height.

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Concerning the same problem, an attack with the indirect methods of the calculus of variations was attempted in references 4 to 6. The results obtained, however, are incomplete. In fact, while the true solution is discontinuous and generally formed of three different branches (dive; zoom; variable path inclination subarc), the authors of these references detected only one of the three subarcs forming the extremal arc, namely the subarc flown with variable path inclination. Complete variational solutions, on the other hand, were detected by Cicala and Miele (ref. 7) and by Lush (ref. 8). In recent times, the results which Miele indicated in reference 1 were rederived by Cartaino and Dreyfus (ref. 9) along the lines of the promising theory of dynamic programming, as developed by Bellman (ref. 10).

Simplified Climbing Problem for Rocket-Powered Aircraft

For a rocket-powered aircraft the timewise variation of mass is important and must be accounted for in the study of the climbing performances. Historically, it is of interest to notice that the theoretical development of new climbing techniques for rocket-powered aircraft preceded that for turbojet-powered aircraft.

Of particular interest are two papers by Kaiser (ref. 11) and Lippisch (ref. 12). Both works were carried out in connection with the pioneering development of jet-propelled aircraft in Germany during World War II. Even though they left the bulk of variational questions associated with the climbing flight substantially unsolved, they threw considerable light on a new class of problems of the mechanics of flight.

In particular, it seems that in the work by Kaiser (ref. 11) the concept of energy height $\left(h_e = h + \frac{V^2}{2g}\right)$ was first employed for flight mechanics applications. Lippisch (ref. 12), on the other hand, investigated the accelerated climbing flight under the simplifying assumption that the total drag coefficient C_D is constant along the flight path.

In recent years the fixed end-points problem was reinvestigated by Miele (ref. 13), under the assumption that the induced drag is negligible with respect to the zero-lift drag. Miele's treatment was based on Green's theorem, in order to avoid some of the analytical difficulties (solved in later works) associated with the use of the indirect methods of the calculus of variations.

Investigations of a More General Nature

The investigations of the sections on climbing flight of minimum time or minimum fuel consumption were carried out under particular hypotheses, whose essential analytical objective was to simplify the equation of motion on the normal to the flight path. By lifting the above limitations, a more general category of variational problems is originated. These problems (which can be indifferently formulated within the frame of the questions of either the Lagrange, Mayer, or Bolza type) attracted the attention of Cicala and Miele (ref. 14), who made use of the Mayer formulation in an initial note dealing with minimum time flight paths. More general problems of Mayer type were investigated by Cicala in reference 15 and Miele in reference 16.

Problems of the Lagrange type were treated in reference 17, which dealt with brachistocronic paths and in reference 18, which dealt with maximum range trajectories. Both papers, however, must be considered as incorrect. As a matter of fact, the minimal conditions were stated without considering that the equations of motion must be satisfied at all points of the flight trajectory and that, as a consequence, they must appear as nonholonomic constraints in the very formulation of all variational problems of the mechanics of flight.

The Lagrange formulation was also used in reference 19, which dealt with both trajectories of minimum time and of minimum fuel consumption. A parabolic drag polar with coefficients depending upon the Mach number was assumed. After eliminating the lift from the equations of motion, the nonholonomic constraint to be satisfied at all points of the flight path was formulated in the form

$$f(h,V,\dot{V},\theta,\dot{\theta},m) - 1 = 0$$

This formulation, however, is not as general as the one of reference 14, which is valid for all drag polars.

A recent paper by Fraeijs de Veubeke (ref. 20) refers particularly to the problem of maximum range. After formulating the equations of motion in parametric form, the paper considers several possible simplifications in the analytical nature of the drag function.

Object of Present Investigation

With the present investigation the work initially developed in references 14, 15, and 21 is extended. General equations are presented describing the optimum paths in a vertical plane. These equations are considerably simpler than those of reference 19 and, therefore, more suitable for digital machine computations. The Mayer formulation, used in reference 16, is abandoned here in favor of the perhaps more general formulation due to Bolza. The reason is that the Bolza formulation appears to be more flexible than the Mayer one in connection with the simplified analysis of certain special types of flight paths of aircraft propelled by air-breathing engines. Problems of either the Lagrange type or the Mayer type are studied as particular cases.

A novelty of this paper is that the customary form of the equations of motion is modified by the introduction of a special set of constant coefficients K_i $(i = 1 \ . \ . \ . \ 8)$ whose value can either be zero or one, depending upon the particular simplifying hypotheses considered in solving a certain variational problem. In so doing a single set of Euler equations is written, valid for both the case where the exact equations of motion are used and for the case where particular approximations are employed.

Thus, the preliminary analyses carried out by Miele and Cappellari in references 22 and 23 are particular cases of the present theory obtained by merely attributing to the coefficients K_i $(i = 1 \ . \ . \ . \ 8)$ the set of values which corresponds to the form of the equations of motion.

Furthermore, a bridge is established between calculus of variations and ordinary theory of maxima and minima. It is shown that, when the acceleration terms are neglected, both approaches lead to the same results (refs. 24 and 25). In this connection, the cruising flight of maximum range or of maximum endurance is investigated for constant altitude or for constant value of the control parameter of a turbojet engine. Simple relations are obtained for the optimizing condition. Thus, the well-known results of the low-speed flight theory are extended to cover the case where an arbitrary dependence between zero-lift drag coefficient, induced drag coefficient, thrust, specific fuel consumption, and Mach number is considered.

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SYMBOLS

a	speed of sound, ft sec^{-1}
A_1	function defined by equation (91)
A_2	function defined by equation (92)
A_3	function defined by equation (93)
B_1	function defined by equation (105)
B_2	function defined by equation (106)
B_3	function defined by equation (107)
c	specific fuel consumption per unit time and unit thrust, sec^{-1}
C	integration constant
C_D	drag coefficient
C_L	lift coefficient
D	drag, lb
F	fundamental function defined by equation (14)
g	acceleration of gravity, ft sec^{-2}
G	functional form whose difference ΔG is to be minimized in a Mayer problem
h	flight altitude above sea level, ft
H	function whose time integral is to be minimized in a Lagrange problem
J	functional form to be minimized in a Bolza problem
$J_1 \ldots J_8$	first members of the equations representing constraints of the variational problem
K	ratio of induced drag coefficient to square of lift coefficient

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$K_1 \ldots K_8$	dimensionless constants
L	lift, lb
m	instantaneous mass of aircraft, lb ft^{-1} sec^2
M	Mach number, V/a
p	atmospheric pressure, lb ft^{-2}
R	ratio of induced drag to zero-lift drag, $K{C_L}^2/C_{Do}$
S	reference surface, ft^2
t	time, sec
T	thrust, lb
V	flight velocity, ft sec^{-1}
V_e	equivalent exit velocity of rocket engine, ft sec^{-1}
x	exponent appearing in expression for zero-lift drag coefficient
X	horizontal distance, ft
y	coefficient appearing in expression for zero-lift drag coefficient
$z_1 \ldots z_9$	generic dependent variables
α	control parameter of jet engine
β	engine mass flow, lb ft^{-1}sec
γ	ratio of specific heat at constant pressure to specific heat at constant volume
$\delta(\ldots)$	variation consistent with prescribed end conditions
η	real variable defined by equation (11), $lb^{1/2}$
θ	path inclination with respect to horizontal plane
λ	dimensionless parameter proportional to wing loading of aircraft

$\lambda_1 \ldots \lambda_8$ Lagrange multipliers

ξ real variable defined by equation (10), $lb^{1/2}$

π ratio of static pressure at altitude h to static pressure at the tropopause, p/p_*

τ dimensionless parameter proportional to thrust loading of rocket-powered aircraft

Φ first member of the additional constraining equation

Ψ function defined by equation (99)

Superscript:

$(\dot{\ })$ derivative with respect to time

Subscripts:

i initial point

f final point

\- condition immediately before a corner point

\+ condition immediately after a corner point

* condition at altitude of tropopause

FUNDAMENTAL HYPOTHESES AND EQUATIONS OF MOTION

The trajectories considered in the present report are entirely contained in a vertical plane, that is, in a plane perpendicular to the surface of the earth, assumed flat. The vector acceleration of gravity is regarded a constant everywhere.

The airplane is conceived as a particle on which aerodynamic forces of symmetric type are acting. The latter are calculated as in unaccelerated flight, that is, by neglecting the so-called aerodynamic lag. Thus, the relationship between lift L and drag D is assumed to have the general form

$$D = D(h,V,L) \tag{1}$$

The power plant considered here is of a nonspecified type. The only assumption made is that thrust T and mass flow of either fuel or propellant β are specified, though arbitrary, functions of the form

$$T = T(h,V,\alpha) \tag{2}$$

$$\beta = \beta(h,V,\alpha) \tag{3}$$

In the above representation α is a variable defining the operating condition of the engine and can be termed, for instance, control parameter or power setting. Its function is analogous to the one of the accelerator pedal in a typical automobile. In view of the great variety of existing types of engines, it is not convenient to specify the actual physical meaning of α to the effect of developing a general theory. As an example, however, the α variable can be the number of revolutions of the compressor-turbine group of a turbojet engine having constant geometry; or the position of the fuel control lever; or the pressure in the combustion chamber of a rocket engine having fixed geometry. Generally speaking, the control parameter α cannot assume any arbitrary value, but only those values for which the thrust is bounded between an upper limit and a lower limit. Assuming that the latter is ideally zero, the following inequality is to be added to equations (2) and (3) in order to completely define the behavior of the engine

$$0 \leqq T(h,V,\alpha) \leqq T_{max}(h,V) \tag{4}$$

Constraining Equations

After assuming the thrust tangent to the flight path, the following set of equations is considered[2]:

$$J_1 \equiv \dot{X} - V(K_1 + K_2 \cos \theta) = 0 \tag{5}$$

$$J_2 \equiv \dot{h} - V \sin \theta = 0 \tag{6}$$

$$J_3 \equiv K_3\dot{V} + K_4 g \sin \theta + \frac{D(h,V,L) - T(h,V,\alpha)}{m} = 0 \tag{7}$$

$$J_4 \equiv K_5\dot{\theta} + \frac{g}{V}\left(K_6 + K_7 \cos \theta\right) - \frac{L}{mV} = 0 \tag{8}$$

$$J_5 \equiv \dot{m} + K_8\beta(h,V,\alpha) = 0 \tag{9}$$

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[2] The constant coefficients K_1, K_2 . . . K_8 are introduced to facilitate the derivation of simplified solutions, under particular hypotheses. The exact equations of motion are associated with the following set of values: $K_1 = K_6 = 0$, $K_2 = K_3 = K_4 = K_5 = K_7 = K_8 = 1$.

The inequality (4) can be translated into the setting of the variational problem, by introducing two new real variables ξ and η satisfying the further constraints

$$J_6 \equiv T(h,V,\alpha) - \xi^2 = 0 \tag{10}$$

$$J_7 \equiv T_{max}(h,V) - T(h,V,\alpha) - \eta^2 = 0 \tag{11}$$

Clearly, the effect of the constraints (10) and (11) is to reconduce the analytical treatment of a variational problem involving inequalities to the same mathematical model useful in solving problems involving equalities.

Additional Constraint

The set of seven equations (5) to (11) involves one independent variable, namely the time t. There are nine dependent variables X, h, V, θ, m, L, α, ξ, and η. For the case where the thrust is tangent to the flight path, therefore, the problem of the optimum trajectory embodies 2 degrees of freedom.

Problems of a more limited nature (involving 1 degree of freedom) can also be of engineering interest. These problems arise in those cases where an additional constraint of the form

$$J_8 \equiv \Phi(X,h,V,\theta,m,L,\alpha,t) = 0 \tag{12}$$

is considered. As an example, a rectilinear path is represented by $\Phi \equiv \theta$ - Const.; a nonlifting path by $\Phi \equiv L$; a path flown with constant value of the engine control parameter by $\Phi = \alpha$ - Const., etc.

PROBLEM OF BOLZA

The functional form

$$J \equiv \Big[G(X,h,V,\theta,m,t)\Big]_i^f + \int_{t_i}^{t_f} H(X,h,V,\theta,m,L,\alpha,t)dt \tag{13}$$

where the terms G and H denote arbitrarily specified functions of the coordinates of the vehicle, is now considered and the problem of Bolza formulated. The latter consists of finding, in the class of arcs $X(t)$, $h(t)$, $V(t)$, $\theta(t)$, $m(t)$, $L(t)$, $\alpha(t)$, $\xi(t)$, and $\eta(t)$ satisfying equations (5) to (12) and certain prescribed end conditions, that special arc such that the functional form (13) is minimized.

To solve the above problem, a set of variable Lagrange multipliers $\lambda_1(t) \ . \ . \ . \ \lambda_8(t)$ is introduced and the following expression, termed fundamental function, formed

$$F = H + \sum_{K=1}^{8} \lambda_K J_K \tag{14}$$

where $J_1 \ . \ . \ . \ J_8$ denote, respectively, the first members of equations (5) to (12). Since the unknown functions are nine in number, nine Euler equations must be written, as follows

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{z}_J}\right) = \frac{\partial F}{\partial z_J} \qquad (J = 1 \ . \ . \ . \ 9) \tag{15}$$

where $z_1 = X$, $z_2 = h$, $z_3 = V$, $z_4 = \theta$, $z_5 = m$, $z_6 = L$, $z_7 = \alpha$, $z_8 = \xi$, and $z_9 = \eta$. The explicit form of the Euler equations is now indicated

$$\dot{\lambda}_1 = \lambda_8 \frac{\partial \Phi}{\partial X} + \frac{\partial H}{\partial X} \tag{16}$$

$$\dot{\lambda}_2 = \frac{\lambda_3}{m}\left(\frac{\partial D}{\partial h} - \frac{\partial T}{\partial h}\right) + \lambda_5 K_8 \frac{\partial \beta}{\partial h} + \lambda_6 \frac{\partial T}{\partial h} + \lambda_7\left(\frac{\partial T_{max}}{\partial h} - \frac{\partial T}{\partial h}\right) + \lambda_8 \frac{\partial \Phi}{\partial h} + \frac{\partial H}{\partial h} \tag{17}$$

$$\begin{aligned} K_3\dot{\lambda}_3 = {} & -\lambda_1\left(K_1 + K_2 \cos \theta\right) - \lambda_2 \sin \theta + \frac{\lambda_3}{m}\left(\frac{\partial D}{\partial V} - \frac{\partial T}{\partial V}\right) + \frac{\lambda_4}{V^2} \frac{L}{m} - \\ & \frac{\lambda_4 g}{V^2}\left(K_6 + K_7 \cos \theta\right) + \lambda_5 K_8 \frac{\partial \beta}{\partial V} + \lambda_6 \frac{\partial T}{\partial V} + \lambda_7\left(\frac{\partial T_{max}}{\partial V} - \frac{\partial T}{\partial V}\right) + \\ & \lambda_8 \frac{\partial \Phi}{\partial V} + \frac{\partial H}{\partial V} \end{aligned} \tag{18}$$

$$\begin{aligned} K_5\dot{\lambda}_4 = {} & \lambda_1 V K_2 \sin \theta - \lambda_2 V \cos \theta + \lambda_3 K_4 g \cos \theta - \lambda_4 \frac{g}{V} K_7 \sin \theta + \lambda_8 \frac{\partial \Phi}{\partial \theta} + \\ & \frac{\partial H}{\partial \theta} \end{aligned} \tag{19}$$

$$\dot{\lambda}_5 = \frac{\lambda_3}{m^2}(T - D) + \lambda_4 \frac{L}{m^2 V} + \lambda_8 \frac{\partial \Phi}{\partial m} + \frac{\partial H}{\partial m} \tag{20}$$

$$0 = \frac{\lambda_3}{m} \frac{\partial D}{\partial L} - \frac{\lambda_4}{mV} + \lambda_8 \frac{\partial \Phi}{\partial L} + \frac{\partial H}{\partial L} \tag{21}$$

$$0 = -\frac{\lambda_3}{m} \frac{\partial T}{\partial \alpha} + \lambda_5 K_8 \frac{\partial \beta}{\partial \alpha} + \left(\lambda_6 - \lambda_7\right) \frac{\partial T}{\partial \alpha} + \lambda_8 \frac{\partial \Phi}{\partial \alpha} + \frac{\partial H}{\partial \alpha} \tag{22}$$

$$0 = -2\lambda_6 \xi \tag{23}$$

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$$0 = -2\lambda_7 \eta \tag{24}$$

The following general remarks are of interest: (1) If the H function is identically zero, the Bolza problem reduces to the Mayer problem. If, on the other hand, the G function is identically zero, then the Bolza problem reduces to the Lagrange problem (ref. 26); (2) if the Φ function is not identically zero, the variational problem admits one degree of freedom; if, on the other hand, the Φ function is identically zero, then the number of degrees of freedom is increased to 2; (3) an important mathematical consequence of the set of equations (16) to (24) is that

$$\frac{d}{dt}\left(\lambda_1 \dot{X} + \lambda_2 \dot{h} + K_3 \lambda_3 \dot{V} + K_5 \lambda_4 \dot{\theta} + \lambda_5 \dot{m} - H\right) + \lambda_8 \frac{\partial \Phi}{\partial t} + \frac{\partial H}{\partial t} = 0 \tag{25}$$

(4) generally speaking, the set of Euler equations (16) to (24) and of constraining equations (5) to (12) must be dealt with by approximate integration procedures. The main difficulty stems from the circumstance that the variational problems of the mechanics of flight are boundary-value problems, that is, problems with conditions prescribed in part at the initial point and in part at the final point. Thus, the use of trial-and-error techniques is, in the general case, an unavoidable necessity. At any rate, the present set of equations appears to be much more suitable for digital computing calculations than the set indicated by Behrbohm (ref. 19) or Carstoiu (ref. 27).

Discontinuity of Eulerian Solution

As the Euler equations (23) and (24) show, the extremal arc is discontinuous with regard to the behavior of the engine. In fact, it generally embodies subarcs of three different kinds:

$$\xi = 0, \quad \lambda_7 = 0 \rightarrow T = 0 \tag{26}$$

$$\eta = 0, \quad \lambda_6 = 0 \quad \rightarrow T = T_{max} \tag{27}$$

$$\lambda_6 = 0, \quad \lambda_7 = 0 \tag{28}$$

The subarcs of equation (26) are flown by coasting; those of equation (27) are flown with maximum engine output; and those of equation (28) are generally flown with a continuously variable thrust. The above rule, however, admits several exceptions. An important one is represented by the case where, because of the analytical configuration of the constraint $\Phi = 0$, the behavior of the engine in flight is prescribed at the onset (for instance, constant value of the control parameter α). In such a case, the aforementioned discontinuity disappears.

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When discontinuities are present, the special conditions due to Erdmann and Weierstrass must be satisfied at all corners of the extremal solution. From the theory of reference 26, one may state the above conditions as follows

$$(\lambda_1)_- = (\lambda_1)_+ \tag{29}$$

$$(\lambda_2)_- = (\lambda_2)_+ \tag{30}$$

$$(K_3\lambda_3)_- = (K_3\lambda_3)_+ \tag{31}$$

$$(K_5\lambda_4)_- = (K_5\lambda_4)_+ \tag{32}$$

$$(\lambda_5)_- = (\lambda_5)_+ \tag{33}$$

$$\left(\lambda_1\dot{X} + \lambda_2\dot{h} + K_3\lambda_3\dot{V} + K_5\lambda_4\dot{\theta} + \lambda_5\dot{m} - H\right)_- = \left(\lambda_1\dot{X} + \lambda_2\dot{h} + K_3\lambda_3\dot{V} + K_5\lambda_4\dot{\theta} + \lambda_5\dot{m} - H\right)_+ \tag{34}$$

where the subscript $(\ )_-$ denotes a condition immediately before a corner and the subscript $(\ )_+$ a condition immediately after a corner.

Transversality Condition

The boundary conditions of possible variational problems are in part of the fixed end-points type and in part of the natural type. The latter must be deduced from the following general transversality condition (ref. 26)

$$\Big[\delta G + \lambda_1 \delta X + \lambda_2 \delta h + K_3 \lambda_3 \delta V + K_5 \lambda_4 \delta \theta + \lambda_5 \delta m +$$

$$\left(H - \lambda_1 \dot{X} - \lambda_2 \dot{h} - K_3 \lambda_3 \dot{V} - K_5 \lambda_4 \dot{\theta} - \lambda_5 \dot{m}\right)\delta t\Big]_i^f = 0 \tag{35}$$

The above relation is to be identically satisfied for all systems of variations $\delta(. . .)$ consistent with the prescribed end conditions.

First Integral

If the two functions H and Φ are such that

$$\frac{\partial \Phi}{\partial t} = 0; \ \frac{\partial H}{\partial t} = 0 \tag{36}$$

equation (25) admits the first integral:

$$\lambda_1 \dot{X} + \lambda_2 \dot{h} + K_3 \lambda_3 \dot{V} + K_5 \lambda_4 \dot{\theta} + \lambda_5 \dot{m} - H = C \tag{37}$$

The continuity condition (34) is therefore rewritten as

$$(C)_- = (C)_+ \tag{38}$$

Furthermore, the transversality condition (35) becomes

$$\Big[\delta G + \lambda_1 \delta X + \lambda_2 \delta h + K_3 \lambda_3 \delta V + K_5 \lambda_4 \delta \theta + \lambda_5 \delta m - C\delta t\Big]_i^f = 0 \tag{39}$$

ROCKET TRAJECTORIES FLOWN WITH NEGLIGIBLE INDUCED DRAG

A rocket-powered vehicle is now considered. The control parameter α is chosen identical with the engine mass flow β. Thus, the engine is represented by

$$\beta = \alpha \tag{40}$$

$$T = \alpha V_e \tag{41}$$

$$0 \leqq T \leqq T_{max} \tag{42}$$

where the equivalent exit velocity V_e and the maximum thrust T_{max} are regarded as constant, independent of altitude and velocity.

For $K_1 = K_6 = 0$, $K_2 = K_3 = K_4 = K_5 = K_7 = K_8 = 1$ equations (5) to (9) are rewritten as follows

$$\dot{X} - V \cos \theta = 0 \tag{43}$$

$$\dot{h} - V \sin \theta = 0 \tag{44}$$

$$\dot{V} + g \sin \theta + \frac{D - \alpha V_e}{m} = 0 \tag{45}$$

$$\dot{\theta} + \frac{g \cos \theta}{V} - \frac{L}{mV} = 0 \tag{46}$$

$$\dot{m} + \alpha = 0 \tag{47}$$

If the induced drag is neglected, the total drag D becomes identical with the zero-lift drag D_o. Since D_o depends on altitude and velocity only, one has to write that $D = D(h,V)$. This circumstance leads to the following consequences:

(A) The equation of motion on the tangent to the flight path no longer interacts with the equation of motion on the normal to the flight path. The latter, therefore, can be employed a posteriori (once the variational problem has been solved) to predict the amount of lift necessary to maintain the aircraft on the computed optimum path.

(B) The Euler equation associated with the lift distribution is considerably simplified, insofar as a drag function $D(h,V)$ implies that $\frac{\partial D}{\partial L} = 0$.

Mayer Problem for Constant Mass Flow

If the engine is operated at a constant mass flow (larger than 0, but less than T_{max}/V_e) the additional constraint (12) is written as:

$$\Phi \equiv \alpha - \text{Constant} = 0 \tag{48}$$

Clearly, the two Euler equations (23) and (24) are solved by $\lambda_6 = \lambda_7 = 0$. For a problem of Mayer type $H \equiv 0$ the Euler equation (21) leads to $\lambda_4 = 0$. The remaining Euler equations are written as follows:

$$\dot{\lambda}_1 = 0 \tag{49}$$

$$\dot{\lambda}_2 = \frac{\lambda_3}{m} \frac{\partial D}{\partial h} \tag{50}$$

$$\dot{\lambda}_3 = -\lambda_1 \cos \theta - \lambda_2 \sin \theta + \frac{\lambda_3}{m} \frac{\partial D}{\partial V} \tag{51}$$

$$0 = \lambda_1 V \sin \theta - \lambda_2 V \cos \theta + \lambda_3 g \cos \theta \tag{52}$$

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$$\dot{\lambda}_5 = \frac{\lambda_3}{m^2}\left[V_e \alpha - D\right] \tag{53}$$

$$0 = -\frac{\lambda_3}{m} V_e + \lambda_5 + \lambda_8 \tag{54}$$

Notice that equations (49) to (54) split into 2 noninteracting sets. This is due to the fact that the multipliers λ_5 and λ_8 are present only in equations (53) and (54). The first integral (37) becomes

$$\lambda_1 V \cos \theta + \lambda_2 V \sin \theta + \lambda_3\left(\frac{\alpha V_e - D}{m} - g \sin \theta\right) - \lambda_5 \alpha = C \tag{55}$$

In turn, the transversality condition (39) yields

$$\left[\delta G + \lambda_1 \delta X + \lambda_2 \delta h + \lambda_3 \delta V + \lambda_5 \delta m - C \delta t\right]_i^f = 0 \tag{56}$$

thus indicating that it is not possible, in the present idealized problem, to prescribe the end values for the path inclination θ.

Problems not involving horizontal distance.- If the G function has the form $G = G(h,V,m,t)$ and if no condition is imposed on the X coordinate at one of the two end points, the transversality condition (56) and the Euler equation (49) yield

$$\lambda_1 = 0 \tag{57}$$

everywhere. As a consequence, equation (52) simplifies into (ref. 16)

$$\left(\lambda_2 V - \lambda_3 g\right) \cos \theta = 0 \tag{58}$$

The solution arc is, therefore, discontinuous, being generally composed of vertical subarcs (dive or zoom)

$$\cos\theta = 0 \tag{59}$$

and variable-path inclination subarcs, along which

$$\lambda_2 V - \lambda_3 g = 0 \tag{60}$$

After computing the time derivative of equation (60)

$$\dot{\lambda}_2 V + \lambda_2 \dot{V} - \dot{\lambda}_3 g = 0 \tag{61}$$

one may regard equations (50), (51), (60), and (61) as linear and homogeneous (for $\lambda_1 = 0$) in the multipliers and their derivatives. Nontrivial solutions exist for λ_2, λ_3, $\dot{\lambda}_2$, $\dot{\lambda}_3$ if

$$\begin{vmatrix} 0 & -\frac{1}{m}\frac{\partial D}{\partial h} & 1 & 0 \\ \sin\theta & -\frac{1}{m}\frac{\partial D}{\partial V} & 0 & 1 \\ V & -g & 0 & 0 \\ \dot{V} & 0 & V & -g \end{vmatrix} = 0 \tag{62}$$

that is, if

$$\alpha V_e - D - V\frac{\partial D}{\partial V} + \frac{V^2}{g}\frac{\partial D}{\partial h} = 0 \tag{63}$$

Assume now that the zero-lift drag has the form

$$D = \frac{\gamma}{2} C_{Do}(M) pSM^2 \tag{64}$$

Consider, for simplicity, the portion of the flight path which is embedded in the isothermal stratosphere (a = Constant). Define the nondimensional quantities

$$\pi = \frac{p}{p_*} \tag{65}$$

$$\tau = \frac{2\alpha V_e}{\gamma p_* S} \tag{66}$$

where p_* denotes static pressure at the altitude h_* of the tropopause. From equations (63) to (66) one obtains (ref. 22)

$$\pi = \frac{\tau}{C_{Do}M^2\left[3 + \gamma M^2 + \dfrac{d \log C_{Do}}{d \log M}\right]} \tag{67}$$

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If the relationship between zero-lift drag coefficient and Mach number is approximated as

$$C_{Do} = yM^{x-2} \tag{68}$$

(y and x being appropriate constants) equation (67) modifies into

$$\pi = \frac{\tau}{yM^x\left(1 + x + \gamma M^2\right)} \tag{69}$$

Notice that the results expressed by equations (63), (67), and (69) are valid for all Mayer problems of the form $G = G(h,V,m,t)$. In particular, they hold for the following cases: (1) Brachistocronic climbing technique $G = t$; (2) climbing technique of maximum velocity increase $G = -V$; (3) climbing technique of maximum altitude increase $G = -h$.

Notice also that, for a constant engine mass flow, the flight time is proportional to the propellant consumption. As a consequence the brachistocronic climbing maneuver is identical with the climbing maneuver which minimizes the propellant consumption.

APPROXIMATE COMPUTATION OF FLIGHT PATHS OF TURBOJET-POWERED AIRCRAFT BY NEGLECTING CENTRIPETAL ACCELERATIONS

An interesting approach to the optimum climbing trajectory of a turbojet-powered aircraft can be developed by means of the following simplifications: (1) Centripetal accelerations are neglected; (2) changes in aircraft weight due to fuel consumption are neglected; (3) weight component on the normal to the flight path is assumed identical with mg.

The above hypotheses are reflected in the following choice for the set of constants appearing in equations (5) to (9):

$$K_1 = K_5 = K_7 = K_8 = 0 \tag{70}$$

$$K_2 = K_3 = K_4 = K_6 = 1 \tag{71}$$

The constraining relationships are, therefore, rewritten as follows:

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$$\dot{X} - V \cos \theta = 0 \tag{72}$$

$$\dot{h} - V \sin \theta = 0 \tag{73}$$

$$\dot{V} + g \sin \theta + \frac{D - T}{m} = 0 \tag{74}$$

$$\frac{1}{V}\left(g - \frac{L}{m}\right) = 0 \tag{75}$$

$$\dot{m} = 0 \tag{76}$$

With regard to the mode of operation of the engine, it is assumed that a conveniently selected control parameter α is constant during flight

$$\Phi \equiv \alpha - \text{Constant} = 0 \tag{77}$$

In view of equation (77), the two Euler equations (23) and (24) are solved by $\lambda_6 = \lambda_7 = 0$. The remaining Euler equations split into two noninteracting sets, the essential one being supplied by equations (16) to (19), rewritten here as follows:

$$\dot{\lambda}_1 = \frac{\partial H}{\partial X} \tag{78}$$

$$\dot{\lambda}_2 = \frac{\lambda_3}{m}\left(\frac{\partial D}{\partial h} - \frac{\partial T}{\partial h}\right) + \frac{\partial H}{\partial h} \tag{79}$$

$$\dot{\lambda}_3 = -\lambda_1 \cos \theta - \lambda_2 \sin \theta + \frac{\lambda_3}{m}\left(\frac{\partial D}{\partial V} - \frac{\partial T}{\partial V}\right) + \frac{\partial H}{\partial V} \tag{80}$$

$$0 = \lambda_1 V \sin \theta - \lambda_2 V \cos \theta + \lambda_3 g \cos \theta + \frac{\partial H}{\partial \theta} \tag{81}$$

Assuming that H is independent of t, the first integral (37) yields

$$\lambda_1 V \cos\theta + \lambda_2 V \sin\theta + \lambda_3\left(\frac{T - D}{m} - g \sin\theta\right) - H = C \tag{82}$$

Furthermore, the transversality condition (39) leads to

$$\left[\delta G + \lambda_1 \delta X + \lambda_2 \delta h + \lambda_3 \delta V - C\delta t\right]_i^f = 0 \tag{83}$$

Mayer Problems Not Involving Horizontal Distance

If $H \equiv 0$, if the G function has the form $G = G(h,V,t)$, and if no condition is imposed on the X variable at one of the two end points, the transversality condition (83) and the Euler equation (78) yield

$$\lambda_1 = 0 \tag{84}$$

everywhere. As a consequence, the Euler equation (81) reduces to equation (58), thus indicating that a discontinuous solution occurs. The extremal arc is composed of subarcs obeying equation (59) and subarcs obeying equation (60), along which

$$\begin{vmatrix} 0 & \frac{1}{m}\left(\frac{\partial T}{\partial h} - \frac{\partial D}{\partial h}\right) & 1 & 0 \\ \sin\theta & \frac{1}{m}\left(\frac{\partial T}{\partial V} - \frac{\partial D}{\partial V}\right) & 0 & 1 \\ V & -g & 0 & 0 \\ \dot{V} & 0 & V & -g \end{vmatrix} = 0 \tag{85}$$

The development of equation (85) leads to

$$\frac{\partial}{\partial V}\left[(T - D)V\right] - \frac{V^2}{g}\frac{\partial}{\partial h}(T - D) = 0 \tag{86}$$

a relation which holds for all problems of the form $G = G(h,V,t)$, regardless of the end conditions. In particular, it holds for the brachistocronic climbing program, for which $G \equiv t$.

Turbojet engine operating in isothermal stratosphere.- It is now assumed that a turbojet-powered aircraft is flying in the isothermal stratosphere[3] and that the engine is operated in such a way that

$$T = \pi T_*(M,\alpha) \tag{87}$$

where T_* denotes thrust at the tropopause and $\pi = \frac{p}{p_*}$. For a parabolic drag polar the drag coefficient C_D is supplied by

$$C_D = C_{Do}(M) + K(M)C_L^{\,2} \tag{88}$$

where K is the so-called induced drag factor. As a consequence, the total drag becomes

$$D = \frac{\gamma}{2} C_{Do} pSM^2 + \frac{2KL^2}{\gamma pSM^2} \tag{89}$$

From equations (86), (87), and (89) considerable manipulations yield the following explicit solution (ref. 23) for the optimum pressure-altitude relationship.

$$\pi = \sqrt{\frac{A_1}{A_2 - A_3}} \tag{90}$$

where

$$A_1 = \frac{\lambda K}{M^2}\left(1 + \gamma M^2 - \frac{d \log K}{d \log M}\right) \tag{91}$$

$$A_2 = \frac{C_{Do} M^2}{\lambda}\left(3 + \gamma M^2 + \frac{d \log C_{Do}}{d \log M}\right) \tag{92}$$

$$A_3 = \frac{T_*}{mg}\left(1 + \gamma M^2 + \frac{\partial \log T_*}{\partial \log M}\right) \tag{93}$$

$$\lambda = \frac{2mg}{\gamma p_* S} \tag{94}$$

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[3]The speed of sound is, therefore, a = Constant everywhere.

Lagrange Problems Not Involving Time and Not Involving Horizontal Distance

Assume now that $G \equiv 0$, $H = H(h,V,\alpha)$ and that both time and horizontal distance are free of choice. The transversality condition (83) and the Euler equation (78) yield

$$\lambda_1 = 0;\ C = 0 \tag{95}$$

while the first integral (82) reduces to

$$\lambda_2 V \sin\theta + \lambda_3\left(\frac{T - D}{m} - g\sin\theta\right) - H = 0 \tag{96}$$

In turn, equation (81) leads once more to a discontinuous solution, that is, to a solution composed of subarcs $\cos\theta = 0$ (dive or zoom) and variable path inclination subarcs

$$\lambda_2 V - \lambda_3 g = 0 \tag{97}$$

Along the latter, the Lagrange multiplier λ_2 takes the form

$$\lambda_2 = \frac{1}{\Psi} \tag{98}$$

where

$$\Psi = \frac{(T - D)V}{Hmg} \tag{99}$$

From equations (79), (98), and (99), the multiplier λ_2 can be eliminated leading to:

$$\frac{\partial\Psi}{\partial V} - \frac{V}{g}\frac{\partial\Psi}{\partial h} = 0 \tag{100}$$

Regardless of the end conditions, the above result holds in all the cases where the functional form to be minimized is of the type

$$J \equiv \int_{t_i}^{t_f} H(h,V,\alpha)dt \tag{101}$$

It holds, in particular, for:

(1) $H \equiv \beta$ (climbing maneuver of minimum fuel consumption);

(2) $H \equiv 1$ (climbing maneuver of minimum time);

(3) $H \equiv V$ (climbing maneuver of minimum increase in curvilinear abscissa associated with the flight path; this maneuver can be regarded as almost identical with the so-called steepest climb, if the inclination θ is, on the average, small).

Turbojet engine operating in isothermal stratosphere.- The problem of minimum fuel consumption is now considered. It consists of minimizing the functional form (101), where

$$H \equiv \beta = \frac{cT}{g} \tag{102}$$

In the above expression c is the specific fuel consumption, the weight of fuel consumed per unit time and unit thrust. In connection with an isothermal stratosphere, a thrust function of the form (87) and a specific fuel consumption function of the form

$$c = c_*(M,\alpha) \tag{103}$$

are assumed. For a parabolic drag polar obeying equation (89), the fundamental equation (100) can be explicitly solved in terms of the pressure ratio, yielding (ref. 23)

$$\pi = \sqrt{\frac{B_1}{B_2 - B_3}} \tag{104}$$

where

$$B_1 = \frac{\lambda K}{M^2}\left[1 + 2\gamma M^2 + \frac{\partial \log\left(\frac{c_* T_*}{K}\right)}{\partial \log M}\right] \tag{105}$$

$$B_2 = \frac{C_{Do}M^2}{\lambda}\left[3 + \frac{\partial \log\left(\frac{C_{Do}}{c_* T_*}\right)}{\partial \log M}\right] \tag{106}$$

$$B_3 = \frac{T_*}{mg}\left(1 - \frac{\partial \log c_*}{\partial \log M}\right) \tag{107}$$

QUASI-STEADY MAYER PROBLEMS FOR CONSTANT ALTITUDE

A quasi-steady problem is defined as any problem in which the inertia terms appearing in the equations of motion are neglected.

It is to be noted that, while a nonsteady point of view is generally indispensable for a correct analysis of the flight paths of rocket-powered vehicles, the quasi-steady approach is still useful in a number of particular cases, such as the study of range and endurance of a turbojet-powered aircraft. In this connection, the following concept is to be stressed: the calculus of variations leads[4] (for $K_3 \to 0$, $K_5 \to 0$) to results which are identical with those offered by the ordinary theory of maxima and minima in the study of the so-called point performances of an aircraft (ref. 24). In this sense, the quasi-steady problems of the Mayer type must be regarded as an analytical attempt to bridge the gap between the more sophisticated variational methods and the less sophisticated techniques of the ordinary theory of maxima and minima.

If the flight path develops at constant altitude, the additional constraint (12) is represented by

$$\Phi \equiv \theta = 0 \tag{108}$$

As a consequence, for $K_1 = K_3 = K_5 = K_6 = 0$, $K_2 = K_4 = K_7 = K_8 = 1$ the equations of motion are rewritten as follows:

$$\dot{X} - V = 0 \tag{109}$$

$$\dot{h} = 0 \tag{110}$$

$$\frac{1}{m}(D - T) = 0 \tag{111}$$

$$\frac{1}{V}\left(g - \frac{L}{m}\right) = 0 \tag{112}$$

$$\dot{m} + \beta = 0 \tag{113}$$

With regard to the Euler equations, the solution $T = 0$ is excluded, because it is not consistent with equation (111). The two remaining possibilities are, therefore, either the solution $T = T_{max}$ or the variable-thrust solution $\lambda_6 = \lambda_7 = 0$. For a Mayer problem

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[4]Analytically, one has to set $K_3 = K_5 = 0$ in equations (7) and (8).

$H \equiv 0$, the latter solution is determined by the following significant set of Euler equations:

$$\dot{\lambda}_1 = 0 \tag{114}$$

$$0 = -\lambda_1 + \frac{\lambda_3}{m}\left(\frac{\partial D}{\partial V} - \frac{\partial T}{\partial V}\right) + \lambda_5 \frac{\partial \beta}{\partial V} \tag{115}$$

$$\dot{\lambda}_5 = \lambda_4 \frac{L}{m^2 V} \tag{116}$$

$$0 = \lambda_3 \frac{\partial D}{\partial L} - \frac{\lambda_4}{V} \tag{117}$$

$$0 = -\frac{\lambda_3}{m}\frac{\partial T}{\partial \alpha} + \lambda_5 \frac{\partial \beta}{\partial \alpha} \tag{118}$$

which admits the first integral

$$\lambda_1 V - \lambda_5 \beta = C \tag{119}$$

The transversality condition (39) is rewritten as

$$\left[\delta G + \lambda_1 \delta X + \lambda_5 \delta m - C\delta t\right]_i^f = 0 \tag{120}$$

Problems Where No Time Condition Is Imposed

If the G function has the form $G = G(X,m)$ and if no time condition is imposed at one of the two end points, the transversality condition (120) leads to

$$C = 0 \tag{121}$$

As a consequence, equations (115), (118), and (119) can be regarded as linear and homogeneous in λ_1, λ_3, and λ_5. Nontrivial solutions exist for the multipliers if

$$\begin{vmatrix} 1 & \frac{1}{m}\left(\frac{\partial T}{\partial V} - \frac{\partial D}{\partial V}\right) & -\frac{\partial \beta}{\partial V} \\ 0 & \frac{1}{m}\frac{\partial T}{\partial \alpha} & -\frac{\partial \beta}{\partial \alpha} \\ V & 0 & -\beta \end{vmatrix} = 0 \tag{122}$$

that is, if

$$\frac{\partial \log T}{\partial \log \alpha}\left[1 - \frac{\partial \log \beta}{\partial \log V}\right] + \frac{\partial \log \beta}{\partial \log \alpha}\left[\frac{\partial \log T}{\partial \log V} - \frac{\partial \log D}{\partial \log V}\right] = 0 \qquad (123)$$

After observing that $\beta = cT/g$, equation (123) is also rewritten as

$$\frac{\partial \log T}{\partial \log \alpha}\left[1 - \frac{\partial \log c}{\partial \log V} - \frac{\partial \log D}{\partial \log V}\right] + \frac{\partial \log c}{\partial \log \alpha}\left[\frac{\partial \log T}{\partial \log V} - \frac{\partial \log D}{\partial \log V}\right] = 0 \qquad (124)$$

leading to

$$\frac{\partial \log(cD)}{\partial \log V} = 1 \qquad (125)$$

for an ideal power plant such that $\partial c/\partial\alpha \cong 0$, $\partial T/\partial\alpha \neq 0$. For a polar obeying equation (89) the logarithmic derivative of the drag with respect to the velocity takes the form

$$\frac{\partial \log D}{\partial \log V} = \frac{2 + \dfrac{d \log C_{Do}}{d \log M} + R\left(-2 + \dfrac{d \log K}{d \log M}\right)}{1 + R} \qquad (126)$$

If, following a transformation of coordinates from the V,h,α space into the M,π,α space, the specific fuel consumption is conceived as $c = c(M,\pi,\alpha)$, equations (125) and (126) yield

$$R = \frac{1 + \dfrac{\partial \log(cC_{Do})}{\partial \log M}}{3 - \dfrac{\partial \log(cK)}{\partial \log M}} \qquad (127)$$

As a consequence, the optimum Mach number is supplied by

$$\frac{\lambda}{\pi} = M^2\sqrt{\frac{C_{Do}}{K}}\sqrt{\frac{1 + \dfrac{\partial \log\left(cC_{Do}\right)}{\partial \log M}}{3 - \dfrac{\partial \log(cK)}{\partial \log M}}} \qquad (128)$$

When the Mach number derivatives of c, C_{Do}, and K are ideally zero, equations (127) and (128) reduce to

$$R = \frac{1}{3} \tag{129}$$

$$M = \sqrt{\frac{\lambda}{\pi}}\sqrt[4]{3\,\frac{K}{C_{Do}}} \tag{130}$$

that is, to the well-known optimum conditions of the quasi-steady flight theory (ref. 28). Notice that the above results hold for all problems of the form $G = G(X,m)$. In particular, they hold for the maximum range problem $G = -X$.

Problems Where No Condition Is Imposed on Horizontal Distance

If the G function has the form $G = G(t,m)$ and if no condition is imposed on the horizontal distance at one of the two end points, the transversality condition (120) and the Euler equation (114) lead to

$$\lambda_1 = 0 \tag{131}$$

everywhere. As a consequence, equations (115) and (118) can be regarded as linear and homogeneous in λ_3 and λ_5. Nontrivial solutions exist for the multipliers if

$$\begin{vmatrix} \frac{1}{m}\left(\frac{\partial D}{\partial V} - \frac{\partial T}{\partial V}\right) & \frac{\partial \beta}{\partial V} \\ \frac{1}{m}\frac{\partial T}{\partial \alpha} & -\frac{\partial \beta}{\partial \alpha} \end{vmatrix} = 0 \tag{132}$$

that is, if

$$\frac{\partial \log T}{\partial \log \alpha}\frac{\partial \log \beta}{\partial \log V} + \frac{\partial \log \beta}{\partial \log \alpha}\left(\frac{\partial \log D}{\partial \log V} - \frac{\partial \log T}{\partial \log V}\right) = 0 \tag{133}$$

After introducing the specific fuel consumption $c = \beta g/T$, equation (133) is rewritten as

$$\frac{\partial \log T}{\partial \log \alpha}\left(\frac{\partial \log c}{\partial \log V} + \frac{\partial \log D}{\partial \log V}\right) + \frac{\partial \log c}{\partial \log \alpha}\left(\frac{\partial \log D}{\partial \log V} - \frac{\partial \log T}{\partial \log V}\right) = 0 \tag{134}$$

leading to

$$\frac{\partial \log(cD)}{\partial \log V} = 0 \tag{135}$$

for an ideal power plant such that $\partial c/\partial\alpha \cong 0$, $\partial T/\partial\alpha \neq 0$. Assume now that the drag polar obeys equation (89) and that the specific fuel consumption is conceived as $c = c(M,\pi,\alpha)$. Equation (135) consequently supplies the optimum ratio of induced drag to zero-lift drag

$$R = \frac{2 + \dfrac{\partial \log(cC_{Do})}{\partial \log M}}{2 - \dfrac{\partial \log(cK)}{\partial \log M}} \tag{136}$$

In turn, the optimum Mach number is to be consistent with the equation

$$\frac{\lambda}{\pi} = M^2 \sqrt{\frac{C_{Do}}{K}} \sqrt{\frac{2 + \dfrac{\partial \log(cC_{Do})}{\partial \log M}}{2 - \dfrac{\partial \log(cK)}{\partial \log M}}} \tag{137}$$

When the Mach number derivatives of c, C_{Do}, and K are ideally zero, the above expressions simplify into

$$R = 1 \tag{138}$$

$$M = \sqrt{\frac{\lambda}{\pi}} \sqrt[4]{\frac{K}{C_{Do}}} \tag{139}$$

leading to the well-known results of the low-speed flight theory (ref. 29). It should be noticed that the results of this section hold for all problems of the form $G = G(t,m)$. As a consequence, they also hold for the maximum endurance problem $G = -t$.

QUASI-STEADY MAYER PROBLEMS FOR CONSTANT CONTROL PARAMETER

In the present section Mayer problems $H \equiv 0$ of the quasi-steady type are investigated, under the assumption that the engine control parameter α is kept constant during flight. Thus, the additional constraint (12) is specified as

$$\Phi \equiv \alpha - \text{Constant} = 0 \tag{140}$$

Furthermore, it is assumed that the slope of the trajectory is so gentle that: (1) Kinematic relationship between horizontal distance, time, and velocity is approximated as $\dot{X} - V = 0$; (2) weight component on the tangent to the flight path is neglected; (3) weight component on the normal to the flight path is identical with mg.

The following set of values is consequently chosen for the numerical constants appearing in equations (5) to (9)

$$K_1 = K_6 = K_8 = 1 \tag{141}$$

$$K_2 = K_3 = K_4 = K_5 = K_7 = 0 \tag{142}$$

so that the constraining equations are rewritten as

$$\dot{X} - V = 0 \tag{143}$$

$$\dot{h} - V \sin \theta = 0 \tag{144}$$

$$\frac{1}{m}(D - T) = 0 \tag{145}$$

$$\frac{1}{V}\left(g - \frac{L}{m}\right) = 0 \tag{146}$$

$$\dot{m} + \beta = 0 \tag{147}$$

With regard to the Euler equations, the conditions $\lambda_6 = \lambda_7 = 0$ lead to the following system

$$\dot{\lambda}_1 = 0 \tag{148}$$

$$0 = \frac{\lambda_3}{m}\left(\frac{\partial D}{\partial h} - \frac{\partial T}{\partial h}\right) + \lambda_5 \frac{\partial \beta}{\partial h} \tag{149}$$

$$0 = -\lambda_1 + \frac{\lambda_3}{m}\left(\frac{\partial D}{\partial V} - \frac{\partial T}{\partial V}\right) + \lambda_5 \frac{\partial \beta}{\partial V} \tag{150}$$

$$0 = \lambda_2 \tag{151}$$

$$\dot{\lambda}_5 = \lambda_4 \frac{L}{m^2 V} \tag{152}$$

$$0 = \lambda_3 \frac{\partial D}{\partial L} - \frac{\lambda_4}{V} \tag{153}$$

$$0 = -\frac{\lambda_3}{m}\frac{\partial T}{\partial \alpha} + \lambda_5 \frac{\partial \beta}{\partial \alpha} + \lambda_8 \tag{154}$$

admitting the first integral

$$\lambda_1 V - \lambda_5 \beta = C \tag{155}$$

The transversality condition is indicated as

$$\left[\delta G + \lambda_1 \delta X + \lambda_5 \delta m - C\delta t\right]_i^f = 0 \tag{156}$$

Problems Where No Time Condition Is Imposed

If the G function has the form $G = G(X,m)$ and if no time condition is imposed at one of the two end points, the transversality condition (156) leads to:

$$C = 0 \tag{157}$$

As a consequence, equations (149), (150), and (155) can be regarded as linear and homogeneous in λ_1, λ_3, and λ_5. Nontrivial solutions exist for the multipliers if

$$\begin{vmatrix} 0 & \frac{1}{m}\left(\frac{\partial T}{\partial h} - \frac{\partial D}{\partial h}\right) & -\frac{\partial \beta}{\partial h} \\ 1 & \frac{1}{m}\left(\frac{\partial T}{\partial V} - \frac{\partial D}{\partial V}\right) & -\frac{\partial \beta}{\partial V} \\ V & 0 & -\beta \end{vmatrix} = 0 \tag{158}$$

that is, if

$$\left(\frac{\partial \log T}{\partial \log h} - \frac{\partial \log D}{\partial \log h}\right)\left(1 - \frac{\partial \log \beta}{\partial \log V}\right) + \frac{\partial \log \beta}{\partial \log h}\left(\frac{\partial \log T}{\partial \log V} - \frac{\partial \log D}{\partial \log V}\right) = 0 \tag{159}$$

After introducing the specific fuel consumption $c = \beta g/T$ and considering the pressure ratio π as a dependent variable in place of the altitude h, equation (159) is rewritten as

$$\frac{\partial \log c}{\partial \log \pi}\left(\frac{\partial \log T}{\partial \log V} - \frac{\partial \log D}{\partial \log V}\right) + \frac{\partial \log T}{\partial \log \pi}\left(1 - \frac{\partial \log c}{\partial \log V} - \frac{\partial \log D}{\partial \log V}\right) +$$

$$\frac{\partial \log D}{\partial \log \pi}\left(\frac{\partial \log c}{\partial \log V} + \frac{\partial \log T}{\partial \log V} - 1\right) = 0 \qquad (160)$$

Turbojet engine operating in isothermal stratosphere.- An ideal aircraft is now considered with thrust T obeying equation (87), specific fuel consumption c obeying equation (103), and drag D obeying equation (89). Reference is made to the isothermal region of the stratosphere α = Const., in which the logarithmic derivative of the drag with respect to the pressure ratio is given by

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$$\frac{\partial \log D}{\partial \log \pi} = \frac{1 - R}{1 + R} \qquad (161)$$

After simple manipulations, equation (160) yields the optimum ratio of induced drag to zero-lift drag

$$R = \frac{2 + \dfrac{\partial \log\left(C_{Do}/T_*\right)}{\partial \log M}}{4 - \dfrac{\partial \log\left(Kc_*{}^2T_*\right)}{\partial \log M}} \qquad (162)$$

so that the optimum Mach number is to be consistent with the equation

$$\frac{2T_*}{\gamma p_* S} = C_{Do}M^2 \frac{6 - \dfrac{\partial \log\left(\dfrac{Kc_*{}^2T_*{}^2}{C_{Do}}\right)}{\partial \log M}}{4 - \dfrac{\partial \log\left(Kc_*{}^2T_*\right)}{\partial \log M}} \qquad (163)$$

For the particular case where the Mach number derivatives of C_{Do}, K, c_*, and T_* are ideally zero, the above relations lead to

$$R = \frac{1}{2} \qquad (164)$$

$$M = \sqrt{\frac{4T_*}{3\gamma p_* SC_{Do}}} \qquad (165)$$

that is, to the well-known formulas of the low-speed flight theory (ref. 28). It is to be emphasized that the results of the present section hold for all problems of the form $G = G(X,m)$ and, therefore, also for the maximum range problem $G = -X$.

Problems Where No Condition Is Imposed on the Horizontal Distance

If the G function has the form $G = G(t,m)$ and if no condition is imposed on the horizontal distance at one of the two end points, the transversality condition (156) and the Euler equation (148) lead to $\lambda_1 = 0$ everywhere. The two Euler equations (149) and (150) yield the following form for the optimizing condition: W 1 1 8

$$\begin{vmatrix} \dfrac{\partial D}{\partial h} - \dfrac{\partial T}{\partial h} & \dfrac{\partial \beta}{\partial h} \\[2ex] \dfrac{\partial D}{\partial V} - \dfrac{\partial T}{\partial V} & \dfrac{\partial \beta}{\partial V} \end{vmatrix} = 0 \tag{166}$$

that is,

$$\left(\frac{\partial \log D}{\partial \log h} - \frac{\partial \log T}{\partial \log h}\right)\frac{\partial \log \beta}{\partial \log V} - \left(\frac{\partial \log D}{\partial \log V} - \frac{\partial \log T}{\partial \log V}\right)\frac{\partial \log \beta}{\partial \log h} = 0 \tag{167}$$

After introducing the specific fuel consumption c and the pressure ratio π, equation (167) is rewritten as

$$\frac{\partial \log c}{\partial \log \pi}\left(\frac{\partial \log T}{\partial \log V} - \frac{\partial \log D}{\partial \log V}\right) - \frac{\partial \log T}{\partial \log \pi}\left(\frac{\partial \log c}{\partial \log V} + \frac{\partial \log D}{\partial \log V}\right) + \frac{\partial \log D}{\partial \log \pi}\left(\frac{\partial \log c}{\partial \log V} + \frac{\partial \log T}{\partial \log V}\right) = 0 \tag{168}$$

Turbojet engine operating in isothermal stratosphere.- After considering equations (87), (89), and (103), equation (168) can be solved in terms of the ratio of induced drag to zero-lift drag:

$$R = \frac{2 + \dfrac{\partial \log\left(C_{Do}/T_*\right)}{\partial \log M}}{2 - \dfrac{\partial \log\left(KT_* c_*^{2}\right)}{\partial \log M}} \tag{169}$$

The optimum Mach number must satisfy the equation

$$\frac{2T_*}{\gamma p_* S} = C_{Do} M^2 \frac{4 - \dfrac{\partial \log\left(\dfrac{Kc_*{}^2 T_*{}^2}{C_{Do}}\right)}{\partial \log M}}{2 - \dfrac{\partial \log\left(KT_* c_*{}^2\right)}{\partial \log M}} \tag{170}$$

When the Mach number derivatives of C_{Do}, K, T_*, c_* are zero, the above expressions reduce to

$$R = 1 \tag{171}$$

$$M = \sqrt{\frac{T_*}{\gamma p_* S C_{Do}}} \tag{172}$$

implying that the optimum operating altitude is identical with the theoretical ceiling of the aircraft (ref. 24). Notice that the present results hold for all problems of the form $G = G(t,m)$ and, therefore, also for the maximum endurance problem $G = -t$.

CONCLUSIONS

A general theory is presented for analyzing minimal maneuvers of high-performance aircraft in a vertical plane.

The Bolza problem consisting of the simultaneous optimization of the angle-of-attack program and of the thrust program is considered. With regard to the general case, the integration of the set of Euler equations and constraining equations is to be performed by approximate methods. In view of the fact that the variational problems of engineering interest are boundary-value problems, the use of trial-and-error procedures is in order.

Under particular assumptions, however, solutions in a closed form can be derived for the optimizing condition. As an example, problems of Mayer type or of Lagrange type are investigated by neglecting either the induced drag or the centripetal acceleration. Particular attention is devoted to the climbing technique of minimum time or of minimum fuel consumption for turbojet-powered aircraft and rocket-powered aircraft. An explicit relationship is obtained between altitude and Mach number, which bypasses the use of the so-called energy-height method.

Mayer problems of the quasi-steady type are also considered in connection with flight paths of maximum range or maximum endurance for turbojet-powered aircraft. The optimizing condition is evaluated for arbitrary dependence between zero-lift drag coefficient, induced drag factor, thrust, specific fuel consumption, and Mach number. An important link is established between calculus of variations and ordinary theory of maxima and minima; it is shown that, for the quasi-steady problem, both approaches lead to the same results.

Purdue University,
Lafayette, Ind., July 25, 1958.

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